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GENERALIZED GENERAL RELATIVISTIC QUANTUM STATIC FIELD EQUATION AND QUANTIZATION OF ENERGY EQUATION

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ABSTRACT
The Hamiltonian and the momentum for full spherical coordinates are related to each of her. Using conventional particle wave duality, a relativistic quantum equation based on energy momentum tensor of generalized general relativity was derived. The solution shows that the energy and angular momentum are quantized. It also predicts graviton and gravity wave existence.

INTRODUCTION
THE HISTORY OF COSMOLOGY
The history of cosmology starts from the beginning of human life on The earth. People observe the sun moving and distributing light from the morning. At night the moon appear on the sky as a golden hemisphere surrounded by glittery beautiful objects. Some of us ask; what is the nature of these astronomical objects? Why some of them move and look brighter than others [1, 2, 3].The accumulated knowledge for these objects led scientists like Kepler to formulate some rules for some of the regular behaviors of these astronomical objects these rules are concerned with the motion of planets around the sun [4, 5]. Later on lsaag Newton discovered that is called, the gravitational field, is responsible for the motion planets around the sun [6].Gravitational field is used to explain the influence of a massive body that spreads in to the space around it the so called a field, which produces a force on another massive body. Very close. [7, 8].Newton Lows of gravitational succeeded in describing the motion of macroscopic objects [9] until the beginning of twentieth century, where Michelson-Morley experiment indicates the violation of the Newton Low of addition of velocity for light [10, 11, and 12].This experiment shows that the speed of light in vacuum is always constant and is completely independent on the motion of the source or observer [13].

Unfortunately Newton gravitational Low suffers from noticeable setbacks. For instance it fails to describe the prehclian of mercury, beside the failure in describing the behavior of quasi-stellar objects and black holes [14, 15].

In 1915, Albert Einstein developed his theory of general relativity, having earlier shown that gravity dos influence light’s motion [16, 17].Einstein’s theory of general relativity (GR) is one of the fundamental physical theories at the present time; it describes a number of gravitational physical phenomena, which agree with astronomical observations [18, 19].Despite these successes GR suffers from being isolated from the main stream of physics. This is since the equation of motion and the energy momentum tensor conservation Lows differs radically from that in other physical theories, which are derived from the action principle. It also suffers from the lack of a full expression for the energy-momentum tensor of the gravitational field [20, 21, 22,].General Relativity Lows can’t easily explain the behavior of exotic astronomical objects, like black holes, pulsars, quasars and neutron stars [23, 24, and 25].For instance it is difficult to explain the large red shift of quasars within the frame work of GR [26, 27, and 28].The behavior of black holes is even more complex. A black holes is a place where gravity is very

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strong that even light can't get out. The gravity is so strong because matter has been squeezed in to a tiny space this can happen when a star is dying. Because not light can get out, people can't see black holes. They are invisible, space telescopes with special tools help find black holes. The special tools can see how stars that are very close to black holes act differently than other stars [29, 30].Black holes can be big or small. Scion tests think the smallest black holes are as large mountain. Another kind of black holes is called staller its mass can be up to 20 times than the mass of the sun. There may be many, many staller mass black holes in Earth's galaxy. Stiller black holes are made when the center of Avery big star falls in upon itself, o collapse. When this happens, it causes a super move. A super move is an exploding star that blasts part of the star in to space. All these gravitational pheromone seems to reed a full quantum gravitational theory as pointed out by many anthers [29,30].Einstein general relativity GR is isolated in its geometrical content from the main stream of physics. This is because there is no well-established quantum gravitational theory. Thus the behavior of black holes and neutron stars can't be explained fully. More over the unification of gravity with other field is not yet achieved. Different attempts were made to find a full quantum gravitational theories [29, 30, 31, 32, 33, 34, 35].In some of them wheeler de Witt quantum wave function is used to describe the universe evolution [36, 37, 38, 39, 40].In to the approaches canonical quantization based on GR is also proposed [41, 42].A quantum model based on quantized general relativity (GGR) is also proposed by some people [43]. The quantum models based on GR does not based on the Hamiltonian which is not defined in GR. The ones based on GGR do not have a wave function and do not dens be isolated bodies. Motivated by the successes of quantum seismological models [44], the aim of this work is devoted to find quantum mode to quantized static field generated by isolated stars. This can describe the behavior of black holes and quasi-stellar and to remove singularity.

THE FULL SPHERICAL QUANTUM GRAVITY EQUATION

In the system of units where $c \neq 1$ the (quantum general equation) becomes

$$\hat{E}\psi = c\hat{P}\psi + f(r)\psi \qquad (1)$$

This equation can be rewritten by taking in to account the fact that in (classical mechanic), the energy is given by:

$$E = \int F.dr = \int m\frac{dv}{dt}.\underline{dr} = m\int d\underline{v}.\frac{dr}{dt}$$

$$E = m\int \underline{v}.d\underline{v} = \frac{mv^2}{2} \qquad (2)$$

But if the system is oscillating the velocity can thus give by

$$v(t) = v_m sin\omega t \qquad (3)$$

Where the effective velocity is given by:

$$v = \frac{v_{max}}{\sqrt{2}} = \frac{v_m}{\sqrt{2}}\ ,\ \ V^2 = \frac{v_m^2}{2} \qquad (4)$$

If one rewrite (5.5.2) to be

$$E = \frac{mv_m^2}{2} \qquad (5)$$

It follows that

$$E = mv^2 \qquad (6)$$

Alternatively for harmonic oscillator

$$E = \frac{1}{2}m\omega^2A^2 = \frac{1}{2}mv_m^2 = m(\frac{V_m}{\sqrt{2}})^2 = mv_e^2 = mv^2 \qquad (7)$$

Thus

$$E = mv^2 = (mv).v = P.v \qquad (8)$$

$$E = P_xV_x + P_yV_y + P_zV_z \qquad (9)$$

For light

$$V_x = V_y = V_z = c \qquad (10)$$

Thus

$$E = cP_x + cP_y + cP_z \qquad (11)$$

In spherical coordinate

$$E = cP_r + cP_\theta + cP_\phi \qquad (12)$$

In view of (12) equation (1) can be written as

$$E = cP_r + cP_\theta + cP_\phi + f(r) \qquad (13)$$

Multiply both sides by ψ to E get

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$$E\psi = cP_r\psi + cP_\theta\psi + cP_\phi\psi + f(r)\psi \qquad (14)$$

Replacing E and P by their corresponding operates one gets

$$\hat{E}\psi = c\hat{P}_r\psi + c\hat{P}_\theta\psi + c\hat{P}_\phi\psi + f(r) \qquad (15)$$

But the energy and momentum operators take the form

$$\hat{E} = i\frac{\partial}{\partial t}$$
$$\hat{P}_r = \frac{\hbar}{i}\frac{\partial}{\partial t}$$
$$\hat{P}_\theta = -\frac{\hbar}{ir}\frac{\partial}{\partial\theta}\hat{P}_\phi = \frac{\hbar}{i}\frac{1}{rsin\theta}\frac{\partial}{\partial\phi} \qquad (16)$$

Thus the full quantum equation becomes

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{\hbar}{i}\left[c\frac{\partial\psi}{\partial r} + \frac{c}{r}\frac{\partial\psi}{\partial\theta} + \frac{c}{rsin\theta}\frac{\partial\psi}{\partial\phi} + f(r)\psi\right]$$
$$i\hbar\frac{\partial\psi}{\partial t} = \frac{\hbar}{i}\left[c\frac{\partial\psi}{\partial r} + \frac{c}{r}\frac{\partial\psi}{\partial\theta} + \frac{c}{rsin\theta}\frac{\partial\psi}{\partial\phi} + f(r)\psi\right] \qquad (17)$$

Using separation of variables

$$\psi(r,\theta,\phi,t) = \tilde{R}(r) + \theta\Phi(\phi)\,v(t) \qquad (18)$$

Equation (17) becomes

$$iR\Phi\Theta\frac{dv}{dt} = \frac{\hbar c}{i\times i}vH\Theta\Phi\frac{\partial\ddot{R}}{\partial r} + \frac{\hbar c}{i\times i}v\tilde{R}\Phi\frac{\partial\Theta}{\partial\theta} + \frac{c\hbar}{irsin\theta}\tilde{R}\Theta\frac{\partial\Phi}{\partial\phi} + f(r)R\Theta\Phi v$$

Divide by $R\,\Theta\Phi v$

$$\frac{i\hbar}{v}\frac{dv}{\partial t} = \frac{c\hbar}{i}\frac{1}{\tilde{R}}\frac{\partial\tilde{R}}{\partial r} + \frac{c\hbar}{i}\frac{1}{r\Theta}\frac{\partial\Theta}{\partial\theta} + f(r) + \frac{c\hbar}{i}\frac{1}{r\phi sin\theta}\frac{\partial\Phi}{\partial\phi} = C_0 = E$$

Thus

$$\frac{i\hbar}{v}\frac{\partial v}{\partial t} = E \qquad (19)$$

$$f(r) + \frac{c\hbar}{i}\frac{1}{\tilde{R}}\frac{d\tilde{R}}{\partial r} + \frac{ch}{i}\frac{1}{r\Theta}\frac{\partial\Theta}{\partial\theta} + \frac{ch}{i\phi(crsin\theta)}\frac{\partial\Phi}{\partial\phi} = E$$

Multiply by r

$$\frac{c\hbar}{i}\frac{d\tilde{R}}{dr} + f(r)r + \frac{c\hbar}{i\Theta}\frac{d\Theta}{d\theta} + \frac{ch}{\phi(isin\theta)}\frac{d\Phi}{d\phi} = Er$$
$$\frac{c\hbar}{i}\frac{r}{\tilde{R}} + (f(r) - E)r = \frac{ch}{i}\frac{d\Theta}{d\theta} + \frac{ch}{\Phi i}\frac{1}{sin\theta}\frac{d\Phi}{d\phi} = C_1$$

Thus

$$\frac{ch}{i}\frac{r}{\tilde{R}}\frac{d\tilde{R}}{dr} + [f(r) - E]r = C_1 \qquad (20)$$
$$\frac{c\hbar}{i\Theta}\frac{d\Theta}{d\theta} + \frac{ch}{\Phi i}\frac{1}{sin\theta}\frac{d\Phi}{d\phi} = C_1 \qquad (21)$$

Multiply by $(sin\theta)$

$$\frac{c\hbar}{i}\frac{sin\theta}{\Theta}\frac{d\Theta}{d\theta} + \frac{ch}{\Phi i}\frac{d\Phi}{d\phi} = C_1 sin\theta \qquad (22)$$
$$c\hbar\frac{sin\theta}{i\Theta}\frac{d\Theta}{d\theta} - C_1 sin\theta = -\frac{ch}{\Phi i}\frac{d\Phi}{d\phi} = C_1 \qquad (23)$$

Thus

$$\frac{c\hbar}{i}\frac{sin\theta}{\Theta}\frac{d\Theta}{d\theta} - C_1 sin\theta = C_2 \qquad (24)$$

The function $f(r)$can be found from equations (3),(4) where the GGR Hamiltonian is given by

$$H = \alpha R^2 + \alpha\frac{\dot{B}}{AB}\dot{R} = \frac{1}{3}\alpha R^2 + \frac{1}{3}\alpha R^2 + \frac{1}{3}\alpha R^2 + \alpha\frac{\dot{B}}{AB}\dot{R} \qquad (25)$$

The corresponding momentum components in spherical coordinates is given by

$$\frac{1}{3}P_r = \frac{1}{3}\alpha R^2 + \frac{2}{3}\frac{\alpha\ddot{R}}{A} - \frac{\alpha}{3}\frac{\dot{A}\dot{R}}{R^2}$$

$$\frac{1}{3}P_\theta = \frac{1}{3}\alpha R^2 + \frac{\dot{R}}{3rA}\frac{1}{3}P_\phi = \frac{1}{3}\alpha R^2 + \frac{\dot{R}}{3rA} \qquad (26)$$

Thus inserting equation (26) in equation (25) yields

$$H = \frac{1}{3}\left[P_r + P_\theta + P_\phi\right] - \frac{2}{3}\frac{\alpha\ddot{R}}{A} + \frac{\alpha}{3}\frac{\dot{A}}{A^2}\dot{R} - \frac{2}{3}\frac{\dot{R}}{rA} + \frac{\alpha\dot{B}}{AB}\dot{R} \ (27)$$

In the system of units where $c \neq 1$

The Hamiltonian becomes

$$H = \frac{c}{3}\left[P_r + P_\theta + P_\phi\right] + f(r) \qquad (28)$$

Or by following Dirac relativistic quantum equation approach

$$H = c\alpha_r P_r + \alpha_\theta P_\theta + \alpha_\phi P_\phi + f(r) \qquad (29)$$

Where

$$f(r) = -\frac{2}{3}\frac{\alpha\ddot{R}}{A} + \frac{\alpha}{3}\frac{\dot{A}}{A^2}\dot{R} - \frac{2}{3}\frac{\dot{R}}{rA} + \frac{\alpha\dot{B}}{AB}\dot{R}$$

$$f(r) = -\frac{2}{3}\frac{\alpha\ddot{R}}{A} + \frac{\alpha}{3}\frac{\dot{A}}{A^2}\dot{R} - \frac{2}{3}\frac{\dot{R}}{rA} + \frac{\alpha\dot{B}}{AB}\dot{R} \qquad (30)$$

SOLUTION OF THE RADIAL PART

Since most of astronomical objects have spherical shape. Therefore it is suitable to use spherical coordinates. In view of equation (20) the radial part is given by

$$\frac{c\hbar}{i}\frac{r}{R}\frac{d\tilde{R}}{dr} + [f(r) - E]r = C_1 \qquad (31)$$

Thus, separation of R and r dependent parts yield

$$\int\frac{d\tilde{R}}{\tilde{R}} = \frac{iC_1}{c\hbar}\int\frac{dr}{r} + \int\frac{i}{c\hbar}\big(E - f(r)\big)dr + C_4$$

$$\ln\tilde{R} = \frac{iC_1}{c\hbar}\ln r + \frac{i}{c\hbar}\left[Er - \int f(r)dr\right] + C_4 \ln\tilde{R} - \ln r^{ic_3} = \frac{i}{c\hbar}\left[Er - \int f(r)dr\right] + C_4$$

$$\tilde{R} = C_5 r^{ic_3} e^{\frac{i}{c\hbar}[Er - \int f(r)dr]} \qquad (32)$$

Where

$$C_3 = \frac{C_1}{c\hbar}C_5 = e^{c_4} \qquad (33)$$

The radial wave function can be rewritten in the form

$$\tilde{R} = e^{ic_3\ln r} e^{\frac{i}{\hbar}[Er - \int f(r)dr]} \qquad (34)$$

Equation (3) shows that the gravity energy density is constant when

$$R = R_0 = constant \qquad (35)$$

Where

$$H = \alpha R_0^2 = constant \qquad (36)$$

But since the energy density is equation to graviton energy multiplied by the number of them, therefore

$$H = \hbar\omega|\psi|^2 = \hbar\omega\left|\tilde{R}\right|^2 = constant \qquad (37)$$

Therefore the probability or the number of gravitons is also constant, I.e.

$$\left|\tilde{R}\right|^2 = C_6\tilde{R} = C_7 = constant \qquad (38)$$

For simplicity let

$$C_1 = 0 \qquad C_3 = 0 \qquad (39)$$

But since R is constant. Thus according to equation (30)

$$f(r) = 0 \qquad (40)$$

In view of equation (4) and (8)

$$e^{\frac{i}{c\hbar}Er_0} = c_7 \qquad (41)$$

But, since the number of gravitons are constant, and from (6) which hews that $R$is real as for as the Hamiltonian (energy) is real thus

$$\mathrm{R} = \cos\frac{Er_0}{c\hbar} + \mathrm{i}\sin\frac{Er_0}{c\hbar} = c_7\,, \cos\frac{Er_0}{c\hbar} = \ c_7 \sin\frac{Er_0}{c\hbar} = 0 \qquad (42)$$

Hence

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$$\frac{Er_0}{c\hbar} = 2n\pi \qquad E = \frac{2n\pi c\hbar}{r_0} \qquad (43)$$

$$E = \frac{nc\hbar}{r_0} \qquad (44)$$

According to Bohr model the minimum Bohr radian corresponds to

$$2\pi r_0 = \lambda$$

Thus

$$E = \frac{nc\hbar(2\pi)}{\lambda}$$

$$E = 2\pi nf \qquad (45)$$

SOLUTION OF CONSTANT SCALAR CURVATURE

There are many useful solutions which can be found when the mass density is constant. According to GR it implies constant R.using equation (30)

$$f(r) = -\frac{2}{3}\alpha\frac{\ddot{R}}{A} + \frac{\alpha\dot{A}}{3A^2}\dot{R} + \frac{\alpha\dot{B}}{AB}\dot{R} - \frac{2}{3}\frac{\dot{R}}{rA} \qquad (46)$$

For constant R

$$R = R_0 \qquad \dot{R} = 0 \qquad \ddot{R} = 0 \qquad (47)$$

Thus (1) given (2)

$$f(r) = Zero$$

Substituting in (17) and consider ψ to depend on r only for afield generated by a spherical body.

$$\psi = \psi(r) \qquad (48)$$

$$\frac{\partial\psi}{\partial\theta} = Zero \qquad \frac{\partial\psi}{\partial\phi} = Zero \qquad (49)$$

Then

$$i\hbar = \frac{\partial\psi}{\partial t} = \frac{c\hbar}{i}\frac{\partial\psi}{\partial r} + Zero + Zero$$

$$i\hbar = \frac{\partial\psi}{\partial t} = \frac{c\hbar}{i}\frac{\partial\psi}{\partial r} \qquad (50)$$

Substituting

$$\psi = A\sin(kr - \omega t) \qquad (51)$$

$$\frac{\partial\psi}{\partial t} = -\omega A\cos(kr - \omega t)\,\frac{\partial\psi}{\partial r} = kA\cos((kr - \omega t) \ (53)$$

Substituting in

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{hc}{i}\frac{\partial\psi}{\partial r} \qquad (52)$$

$$-i\hbar\omega A\cos((kr - \omega t)\,, \frac{hc}{i}kA\cos((kr - \omega t)$$

$$-i\hbar\omega = ck\frac{\hbar}{i}$$

Multiply by i and divide by $\hbar$

$$W = CK \qquad (53)$$

This indicates that the solution is consistent with the ordinary relation between angular frequency and wave number. Thus

$$\psi = A\sin((kr - \omega t) \qquad (54)$$

Is a solution which indicates that gravitons are travelling waves moving with speed of light. Using the relation

$$cos^2 + sin^2 = 1$$

The probability of existence of particles is

$$|\psi|^2 = A^2cos^2\theta \qquad (55)$$

But

$$\cos 2\theta = cos^2\theta - sin^2\theta = -2cos^2\theta - 1$$

$$cos^2\theta = \frac{1}{2}[cos2\theta + 1]$$

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$h = |\psi|^2 \frac{A^2}{2}[1 + cos2\theta(kr - \omega t)]$ (56)

This means that a gravitational wave can be propagated with wave function

$\psi = Acos(kr - \omega t)$ (57)

The intensity of waves is

$n = \frac{A^2}{2}[1 + \cos 2(kr - \omega)]$

$n = n_0 + \frac{A^2}{2}\cos 2(kr - \omega t)$ (58)

Where

$n_0 = \frac{A^2}{2}$ (59)

The constant form $n_0$can be considered as. A back ground constituting vacuum energy.

GRAVITATIONAL FIELD QUANTIZATION

The gravitational field for spherically symmetric body satisfies the equation

$i\hbar\frac{\partial\psi}{\partial t} = \frac{\hbar c}{i}\frac{\partial\psi}{\partial r}$ (60)

Separating the variables in to time and radial function

$\psi = uv = u(t)v(r)$ (61)

Thus

$i\hbar\frac{v\partial u}{\partial t} = \frac{\hbar c}{i}\frac{u\partial v}{\partial r}$ (62)

Dividing both sides by uv yield

$\frac{i\hbar}{u}\frac{\partial u}{\partial t} = \frac{\hbar c}{i}\frac{1}{v}\frac{\partial v}{\partial r} = C_0 = E$ (63)

Thus

$i\hbar\frac{\partial u}{\partial t} = Eu$ (64)

Which is the energy Eigen function consider now the solution of equation in the form

$u = e^{-i\alpha t}$ (65)

Substituting equation (64) in equation (65) yield

$-i^2\hbar\alpha u = Eu$ (66)

Thus

$E = \hbar\alpha$ (67)

The periodicity condition requires

$u(t + T) = u(t)$ (68)

$e^{-i\alpha(t+T)} = e^{i\alpha t}$ (69)

$e^{-i\alpha T} = \cos\alpha T - i\sin\alpha T = 1$ (70)

Thus

$\cos\alpha T = 1 \qquad \sin\alpha T = 0$ (71)

Hence

$\alpha T = 2n\pi$ (72)

$\alpha = \frac{2n\pi}{T} = 2n\pi f = n\omega$ (73)

In view of equations (66) and (67) the energy is thus given by

$E = \hbar\alpha = n\hbar\omega$ (74)

This means that the energy of gravity field is quantized.

ANGULAR MOMENTUM QUANTIZATION

Equation (23) can be used to find the relation

$-\frac{c\hbar}{i\phi}\frac{d\phi}{\varphi} = C_2$ (75)

$-\frac{\hbar}{c}\frac{d\phi}{\varphi} = \frac{C_2}{c}\phi$ (76)

$$\hat{L}_z \phi = L_z \qquad (77)$$

Where the L_z operator is given by

$$\hat{L}_z = -\frac{\hbar}{i}\frac{\partial}{\partial \varphi} \qquad (78)$$

$$\int \frac{d\phi}{\phi} = -\frac{i}{\hbar}\frac{C_2}{c}\int d\varphi \qquad (79)$$

$$\ln \phi = -\frac{iC_2}{c\hbar} = -iC_0\varphi + C_3 \qquad (80)$$

$$\phi = e^{c_3} e^{-ic_0\varphi} = \phi_0 e^{-ic_0} \qquad (81)$$

$$\phi(\varphi + 2\pi) = \phi(\varphi) \qquad (82)$$

$$e^{-ic_0(\varphi+2\pi)} = e^{ic_0\varphi} \qquad (83)$$

$$e^{-2\pi c_0 i} = \cos 2\pi c_0 + i \sin 2\pi c_0 = 1 \qquad (84)$$

Thus

$$\cos 2\pi c_0 = 1 \qquad \sin 2\pi c_0 = 0 \qquad (85)$$

$$2\pi C_0 = 2n\pi \qquad (86)$$

$$n = o, 1, 2, 3, \dots \dots$$

$$C_0 = n$$

$$L_z = \frac{C_2}{c} = \frac{\hbar c}{c} C_0 = \hbar n \qquad (87)$$

A full GGR quantum equation for spherically symmetric motion is obtained in equation (17). The separation of variables (18) leads to 4 independent (r, θ, ϕ, t) equations. The time dependent and radial parts [see equation (5)] predict. Again the existence of gravitational wave. Applying time periodicity of this wave on equation (15). Again this energy is nothing but quantum plank ordinary energy. The angular.ϕPart of Schrödinger equation is given by equation (1). Its solution is given by equation (7). Using the fact that the wave function has only one unique value, the angular momentum component L_z is shown to be quantized.

CONCLUSION

The GGR quantum equation is useful in quantizing energy and angular momentum component along z direction. It predicts the existence of graviton and gravitational Waves.

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[40] section 5 in ch. 12 of Weinberg 1972
[41] Introductory chapters of Stephani et al. 2003
[42] A review showing Einstein's equation in the broader context of other PDEs with physical significance is Geroch 1996
[43] For background information and a list of solutions, cf. Stephani et al. 2003; a more recent review can be found in MacCallum 2006
[44] Chandrasekhar 1983, ch. 3,5,6
[45] Narlikar 1993, ch. 4, sec. 3.3